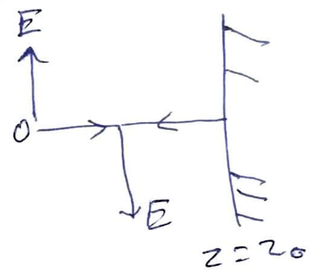
Cavity → a rectangular wave guide shorted at both ends.

Phenomena similar to that of standing wave patterns on a string occur in electromagnetic resonance

Consider a plane wave with components E_x and H_y, travelling in the z-direction.
Suppose a perfect conductor is placed into the half infinite space $z > z_0$



Since the wave incident on the conductor turns back, the total electric field is given by

$$E_x = A_i e^{i(kz-\omega t)} + A_r e^{-i(kz+\omega t)}$$

This has to satisfy the boundary condition

$$E_x = 0 \text{ at } z = z_0$$

Therefore

$$A_i e^{i(kz_0-\omega t)} + A_r e^{-i(kz_0+\omega t)} = 0$$

i.e. $$A_r = -A_i e^{2ikz_0}$$

$$\Rightarrow E_x = A_i \left\{ e^{i(kz-\omega t)} - e^{2ikz_0} e^{-i(kz+\omega t)} \right\}$$

$$= A_i e^{-i\omega t} \left\{ e^{ikz} - e^{2ikz_0} e^{-ikz} \right\}$$

$$= A_i e^{-i\omega t} e^{ikz_0} \left\{ e^{ik(z-z_0)} - e^{-ik(z-z_0)} \right\}$$

$$= 2i A_i e^{-i\omega t} e^{ikz_0} \sin k(z-z_0) \quad \text{—(1)}$$

which → represent a standing wave

The standing wave pattern does not undergo any change If an infinitely conducting plate is placed parallel to the first plate at the position of a node where the electric field is zero.

The two plates, thus, constitute a resonator in which the electromagnetic energy bounces between the plates

Consider a closed box formed by placing end faces on a rectangular wave guide.

We will assume the end surfaces are plane and perpendicular. Because of the reflection at the end faces, the waves in the cavity are standing waves and not progressive.

This cavity resonator will resonate at a frequency at which the length of cavity is an integral multiple of the half-wavelength measured in the wave guide.

The electric field components are of the form

$$\left.\begin{aligned} E_x &= E_1 \cos(k_1 x)\, \sin(k_2 y)\, \sin(k_3 z)\, e^{-i\omega t} \\ E_y &= E_2 \sin(k_1 x)\, \cos(k_2 y)\, \sin(k_3 z)\, e^{-i\omega t} \\ E_z &= E_3 \sin(k_1 x)\, \sin(k_2 y)\, \cos(k_3 z)\, e^{-i\omega t} \end{aligned}\right\} (2)$$

In order that the boundary conditions be satisfied, it is necessary that k_1, k_2, k_3 have the values

$$k_1 = \frac{l\pi}{a}, \quad k_2 = \frac{m\pi}{b}, \quad k_3 = \frac{n\pi}{c} \quad \text{--- (3)}$$

$a, b, c \rightarrow$ dimensions of the box, $l, m, n \rightarrow$ integers

Substitution of any component in the appropriate wave-eqn shows that the fields given by eqn (2) to be acceptable, the free space wave number has to satisfy the condition

$$k^2 = \frac{\omega^2}{c^2} = \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) \text{--- (4)}$$

$\rightarrow$ infinite number of resonant frequencies and hence infinite number of modes of the cavity corresponding to the different values of l, m, n

The magnetic field components can be found from Maxwell's eqn

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

If $l=0$, $m=1$, $n=1$, the electric field is transverse to the direction of propagation. This mode is designated by TE_{011} mode.

Besides TE_{lmn} modes $\rightarrow$ there are also other modes possible in the cavity

These are TM_{lmn} modes → magnetic field is transverse to the direction of propagation

For the same cavity, TE_{lmn} waves and TM_{lmn} waves occur at the same frequency.

Hence at particular resonant frequency the standing wave in the cavity is sum of two resonating waves, the TE mode and TM mode

Black-body radiation theory → the factor 2 appears in the density of states function.

For a definite field configuration the cavities have some discrete resonant frequencies

⇒ If we try to excite a particular mode of oscillation in a cavity, the right sort of fields will not be built up unless the exciting frequencies is equal to the resonance frequency

Appreciable excitation occurs over a narrow band of frequencies around the resonant frequency.

This smearing out of the sharp frequency of oscillation occurs partly, because of the dissipation of energy in the cavity walls.
The measure of these losses are often expressed by Q → quality factor of the cavity

$$Q = \frac{\omega \times \text{Energy stored in the cavity}}{\text{Energy lost per cycle to the walls}}$$

Power loss in the cavity can be estimated by computing the time average of the Poynting vector into the walls at the surface

$$\langle N \rangle = \frac{1}{2} \mathrm{Re} \left(E_{\parallel} \times H_{\parallel} \right)$$

$E_{\parallel}$, $H_{\parallel}$ → tangential components of the electric & magnetic fields

Dispersion relations in Plasma

Plasma → ionized gas consisting of charged particles (e.g. electrons & ions)

Plasma waves:- Various waves can be excited easily in a plasma.

Plasma is nearly charge neutral, so the $\nabla \cdot \vec{E} = 0$ still holds.

However, neither the conduction current or displacement current can be ignored

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Effective Permittivity in a plasma

In vacuum, the phase velocity of an EM wave is

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

phase velocity of waves in matter

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon \mu}}$$

In plasma $\mu = \mu_0$ but $\varepsilon \neq \varepsilon_0$

Electric field, magnetic field, and generally any quantity in plasma can be divided into two parts. DC part that does not depend on time and parts that associated with waves

$$Q_{total} = Q_{DC} + Q(\vec{r}, t)$$

We study only part associated with waves, Q, and assume

$$Q = Q_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

The velocity of electrons (derived → study waves in conductors)

$$\vec{v} = -\frac{e}{m(\nu - i\omega)}\vec{E}$$

$$\vec{J} = \frac{ne^2}{m(\nu - i\omega)}\vec{E}$$

In plasma, the current is predominantly carried by electrons, as in a conductor
↓ mass of electron is smaller than that of ions.

if there is no DC magnetic field

(So we can ignore $\vec{v}\times\vec{B}$ term in the eq^n of motion)

In most cases, the electron density, and the collision frequency ν, are much smaller than in conductors, so

$$\vec{J} \simeq -\frac{ne^2}{mi\omega}\vec{E} = i\frac{ne^2}{m\omega}\vec{E}$$

$$\Rightarrow \nabla\times\vec{B} = \mu_0\left(i\frac{ne^2}{m\omega}\vec{E} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$= \mu_0\left(i\frac{ne^2}{m\omega}\vec{E} - i\omega\varepsilon_0\vec{E}\right)$$

$$= -i\omega\varepsilon_0\mu_0\left(1 - \frac{ne^2}{m\varepsilon_0\omega^2}\right)\vec{E}$$

Let $\omega_p = \sqrt{\frac{ne^2}{m\varepsilon_0}}$ → plasma frequency

and recall $\vec{H} = \vec{B}/\mu_0$, we find

$$\nabla\times\vec{H} = -i\omega\varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)\vec{E}$$

For low frequency wave, $\omega_p \gg \omega$, conduction current dominates (like waves in a conductor)

For high frequency wave, $\omega_p \ll \omega$, displacement current dominates (like waves in vacuum).

If we define an effective permittivity in plasma

$$\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

The 4th Maxwell's eq^n for a monochromatic plane wave in plasma becomes

$$\nabla\times\vec{H} = -i\omega\varepsilon\vec{E} = \varepsilon\frac{\partial\vec{E}}{\partial t}$$

Similar as monochromatic plane wave in vacuum, except $\varepsilon_0 \to \varepsilon$.

Effective permittivity in plasma depends on the frequency ω. $\varepsilon < \varepsilon_0$

Dispersion Relation

The phase velocity of plasma waves:

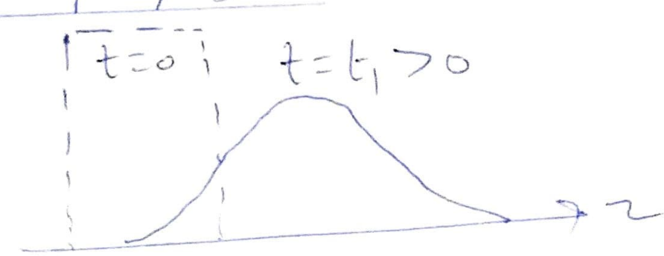
$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$= \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \frac{1}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$$

$$= \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} > c \quad \text{---(1)}$$

In plasma, ε depends on ω and thus the phase velocity also depends on ω. The wave in this case is dispersive

A square wave contains the fundamental frequency and its higher harmonics. If the phase velocity depends on frequency, it will spread out while it propagates



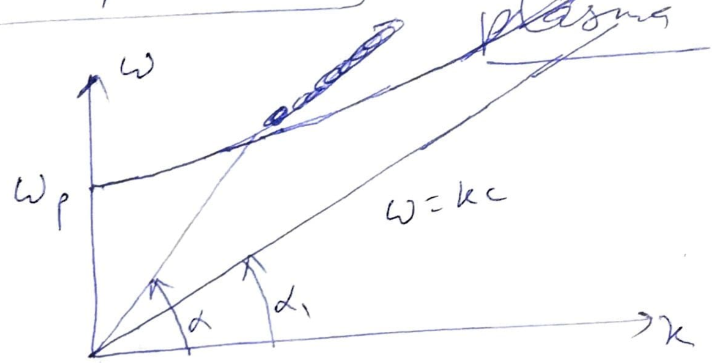
The dependence of ω on k is called dispersion relation

From eq^n (1), for ω

$$\frac{\omega^2}{k^2} = \frac{c^2}{1-\frac{\omega_p^2}{\omega^2}}$$

$$\omega^2\left(1-\frac{\omega_p^2}{\omega^2}\right) = c^2k^2$$

$$\boxed{\omega^2 = \omega_p^2 + (ck)^2}$$ dispersion relation in plasma



The dispersion relation of EM waves in vacuum is a straight line (non-dispersive) with a slope $\tan\alpha = c$

The dispersion relation of EM waves in plasma is above the line $\omega = ck$ because

$$\omega = \sqrt{\omega_p^2 + (ck)^2} \geq ck$$

but approaches the line $\omega = ck$ when k becomes larger because

$$\omega = \sqrt{\omega_p^2 + (ck)^2} \simeq ck \quad \text{when } k \gg \frac{\omega_p}{c}$$

In plasma, phase velocity

$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1-\frac{\omega_p^2}{\omega^2}}} > c$$

But the phase velocity does not correspond to the ~~f~~ information (energy) propagation velocity. The information is propagating at the group velocity

$$v_g = \frac{d\omega}{dk}$$

For non dispersive waves, i.e. EM waves in vacuum

$$\omega = ck \qquad v_p = v_g = c$$

In plasma, wave is dispersive. The group velocity is

$$v_g = \frac{d\omega}{dk} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} < c$$